Modal Quantification over Structured Domains

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1 Quantifiers as Modal Operators

1.1 Motivations

The semantics for quantifiers described in this paper can be viewed both as a new semantics for generalized quantifiers and as a new look at standard first-order quantification, bringing the latter closer to modal logic.

The standard semantics for generalized quantifiers interprets a monadic generalized quantifier Q as a set of subsets of a domain. For example, the quantifier "there are precisely two" is interpreted by the set of all subsets of the domain which contain precisely two elements. A formula $Qx\varphi$ is true in a model if the set of elements satisfying φ belongs to the interpretation of the quantifier; in our example, if there are precisely two elements satisfying φ. The existential quantifier can be treated as a generalized quantifier, too: it is interpreted as the set of all non-empty subsets of the domain. The universal quantifier is interpreted by the singleton set containing the whole domain.

The quantifiers listed so far are first-order definable in the following sense: they can be defined using ordinary quantifiers and equality. Many interesting generalized quantifiers are not first-order definable. The present study is motivated by the work of Michiel van Lambalgen (1991) on Gentzen-style proof theory for the quantifiers "for many" (its dual is interpreted as a non-principal filter), "for uncountably many" and "for almost all" (the latter contains all subsets of the domain which have Lebesgue measure 1). All those quantifiers are not first-order definable. They have Hilbert-style axiomatizations, but until lately no one believed that they can have a reasonable Gentzen-style proof theory. In order to devise such a proof theory, van Lambalgen used a translation of generalized quantifier formulas into a first-order language enriched with a predicate R of indefinite arity. $Qx\varphi(x, y_1, \dots, y_n)$, where Q is a universal-type generalized quantifier (distributing over conjunction), is translated as $\forall x(R(x, y_1, \dots, y_n) \to \varphi(x, y_1, \dots, y_n))$, and its dual Q^d as $\exists x(R(x, y_1, \dots, y_n) \wedge \varphi(x, y_1, \dots, y_n))$ (all free variables displayed). Observe that this translation is reminiscent of the standard translation of modal formulas into first-order logic, with the sequence of free variables playing the role of the "actual world" and the quantifier ranging over the variables "accessible" from the given sequence. The idea behind such a translation is as follows. When generalized quantifiers are viewed as first-order operators (binding first-order variables), it becomes clear that a variable bound by a generalized quantifier cannot in general take any possible value. Its range is restricted, and this restriction can be defined using an accessibility relation. Then the elimination rule for Q with a premise $Qx\varphi(x, \bar{y})$ would introduce a variable $x_{\bar{y}}$ ranging over the set $\{x : R(x, \bar{y})\}$.

It turns out that some quantifier axioms correspond to first-order conditions on R in the following sense: any set of generalized quantifier formulas is consistent with the axiom if and only if the set of translations is consistent with the corresponding first-order condition on R. For example, $Qx\varphi \wedge Qx\psi \to Qx(\varphi \wedge \psi)$ corresponds in this sense to $R(x, \bar{y}\bar{z}) \to R(x, \bar{y})$.

In case such a correspondent exists, it is easier to find side conditions for elimination and introduction rules for the generalized quantifier satisfying the axiom. In the example above, the rule becomes

$$\frac{Qx\varphi(x,\bar{y})}{\varphi(x_{\bar{z}},\bar{y})}$$

where $\bar{y} \subseteq \bar{z}$ and $x_{\bar{z}}$ ranges over the set $\{x : R(x,\bar{z})\}$. Thus, we have a (generalized) modal logic for quantifiers here, which even exhibits some of the standard modal concerns (such as correspondence). Our aim in this paper is to investigate this general modal logic as such.

Another motivation for this enterprise is a modal-style modification of first-order quantification, aimed at obtaining a system which has some nice properties of modal logic not shared by first-order logic. In this sense, our approach relates to the one taken in Nemeti (1993) where non-standard models for first-order logic were introduced, yielding a decidable quantification theory by imposing restrictions on "accessible assignments". Let us rephrase our general idea. The Tarskian truth condition for the existential quantifier reads as follows:

$$M,[\bar{d}/\bar{y}] \models \exists x\varphi(x,\bar{y}) \;\Leftrightarrow\; \exists d \in D : M,[d/x,\bar{d}/\bar{y}] \models \varphi(x,\bar{y})$$

This may be viewed as a special case of a more general schema, when the element d is required in addition to stand in some relation R to $\bar{d}$ - where R is a finitary relation structuring the individual domain D:

$$M,[\bar{d}/\bar{y}] \models \Diamond_x\varphi(x,\bar{y}) \;\Leftrightarrow\; \exists d \in D :\; R(d,\bar{d}) \;\&\; M,[d/x,\bar{d}/\bar{y}] \models \varphi(x,\bar{y})$$

In the above-mentioned work on the generalized quantifiers "many", "uncountably many" and "almost all", R has properties which are common to different independence relations: linear independence in algebra, probabilistic independence, etc. But we can think of more general applications too, with domains being arranged in different levels of accessibility, or with procedures drawing objects in possible dependencies upon one another. One might read $R(d,\bar{e})$ as

- d can be constructed using $\bar{e}$,
- d is not "too far" from the e's,
- after you have picked up e's from the domain without replacing them, d is still available,

et cetera. Ordinary predicate logic then becomes the special case of flat individual domains admitting of "random access", whose R is the universal relation.

This semantics has some clear analogies with modal logic, with an existential generalized quantifier as an existential modality over some domain with, not a binary, but an arbitrary finitary "accessibility relation". As a consequence, we can apply standard ideas concerning modal completeness and correspondence to understand this broader concept of quantification.

Both motivations, generalized quantifiers and generalized first-order semantics, give rise to a variety of questions, including model theory, first-order completeness, canonicity, frame correspondence, definability of R-properties by quantifier axioms etc. In this paper we explore some of them.

1.2 Language and models

The language of the logic $EL(\exists, \Diamond)$ with a generalized quantifier is the ordinary language of first-order predicate logic with equality (without functional symbols) plus an existential generalized quantifier $\Diamond$. The notion of a w.f.f. is extended as follows: if φ is a w.f.f., then so is $\Diamond_x\varphi$. A universal dual of $\Diamond$ is defined as usual: $\Box_x\varphi =_{df} \neg\Diamond_x\neg\varphi$. We shall refer to the sublanguage without ordinary quantifiers as $EL(\Diamond)$.

$M = (D, R, V)$ is a *model* for $EL(\exists, \Diamond)$ if D and V are an ordinary domain and interpretation for first-order logic, and R is a binary relation between $d \in D$ and finite sequences $\bar{d}$ from D. Given the truth definition below, this is equivalent to considering a relation $R(d, D_0)$ between individual objects and finite *sets* D_0 of such objects.

The relation $M, v \models \varphi$ ("φ is *true* in M under assignment v") is defined as follows:

- $M, v \models P_i^n(x_{j1} \dots x_{jn}) \Leftrightarrow < v(x_{j1}) \dots v(x_{jn}) > \in V(P_i^n)$;
- $M, v \models \neg\varphi \Leftrightarrow M, v \not\models \varphi$;
- $M, v \models \varphi \wedge \psi \Leftrightarrow M, v \models \varphi$ and $M, v \models \psi$;
- $M, v \models \exists x\psi(x) \Leftrightarrow$ there exists a variable assignment v' which differs from v at most in its assignment of a value to x ($v' =_x v$) such that $M, v' \models \psi(x)$;
- $M, v \models \Diamond_x\psi(x, y_1, \dots, y_n) \Leftrightarrow$ there exists $v' =_x v$ such that $R(v'(x), v'(y_1), \dots, v'(y_n))$ and $M, v' \models \psi(x, \bar{y})$ where $\bar{y}$ are all (and just the) free variables of $\Diamond_x\psi$ listed in alphabetic order.

  It is easy to see that

- $M, v \models \Box_x\psi(x, \bar{y}) \Leftrightarrow$ if for all $v' =_x v$: $R(v'(x), v'(\bar{y})) \Rightarrow M, v' \models \psi(x, \bar{y})$.

We say that $M \models \varphi$ iff $M, v \models \varphi$ for all variable assignments v.

Let us define a *frame* (analogously to modal logic) $F = (D, R)$ as the underlying structure of a set of models with all possible interpretations of predicate letters. $F, v \models \varphi$ if $M, v \models \varphi$ for all models M on F. The formula φ is (globally) valid in F if, for all v, $F, v \models \varphi$ ("$F \models \varphi$").

This system resembles first-order logic in many respects, but no standard property can be taken for granted any more:

Monotonicity is restricted. Let for all variable assignments v $M, v \models \varphi(x_1, x_2) \to \psi(x_1, x_3)$ and for some assignment v $M, v \models \Diamond_{x_1}\varphi(x_1, x_2)$ (there exists $v' =_{x_1} v$ such that $R(v'(x_1), v(x_2))$ and $M, v' \models \varphi(x_1, x_2)$). It does not follow that $M, v \models \Diamond_{x_1}\psi(x_1, x_3)$, because although $M, v' \models \psi(x_1, x_3)$, it is not necessary that $R(v'(x_1), v(x_3))$ holds. Indeed, the general monotonicity rule

$$\frac{\Sigma \vdash \varphi(x, \bar{y}) \to \psi(x, \bar{z})}{\Sigma \vdash \Diamond_x\varphi(x, \bar{y}) \to \Diamond_x\psi(x, \bar{z}),}$$

with x not free in Σ, is invalid. We can accept only **Restricted Monotonicity**, where φ and ψ have the same free variables.

Extensionality is also restricted. Properties which hold for exactly the same objects, are no longer identical. Consider a property P which holds for a single object a: $\forall x(P(x) \equiv x = a)$. Let $R(a, \emptyset)$ and $\neg R(a, a)$. Then, $\Diamond_x P(x)$ is true and $\Diamond_x x = a$ is false.

Substitution therefore should also be restricted: only formulas with the same parameters can be substituted. We do not have in general that

$$D, R, V, v \models \varphi[\alpha/P] \;\Leftrightarrow\; D, R, V[P := [\alpha]_{M,v}], v \models \varphi.$$

2 Axiomatics and Completeness

We shall now develop the basic deductive calculus for our modal quantifier logic.

Definition 1 *The* minimal logic *for $EL(\exists, \Diamond)$ is a calculus of sequents $\Sigma \vdash \varphi$ satisfying the usual rules for first-order logic, including all Boolean principles, as well as the following quantifier rules:*

Restricted Monotonicity plus Distribution

$$\frac{\Sigma \vdash \varphi(x, \bar{y}) \rightarrow \bigvee_{i=1}^{i=n} \psi_i(x, \bar{y})}{\Sigma \vdash \Diamond_x \varphi(x, \bar{y}) \rightarrow \bigvee_{i=1}^{i=n} \Diamond_x \psi_i(x, \bar{y})}$$

where x is not free in Σ, and free variables are exactly those displayed (only x does not necessarily occur free in ψ_i). The convention here is that an empty disjunction is a falsum, both in the premise and the conclusion.

Alphabetic Variants

$$\vdash \Diamond_x \varphi(x, \bar{y}) \equiv \Diamond_z \varphi(z, \bar{y})$$

where z does not occur (free or bound) in $\varphi(x, \bar{y})$.

Here are some derivations in this system, corresponding to obvious validities given the above existential truth condition for the quantifier $\Diamond$:

1. $\vdash \bot \rightarrow \bot$

   $\vdash \Diamond_x \bot \rightarrow \bot$

   $\vdash \neg \Diamond_x \bot$

2. $\neg\varphi(\bar{y}) \vdash \varphi(\bar{y}) \rightarrow \bot$

   $\neg\varphi(\bar{y}) \vdash \Diamond_x \varphi(\bar{y}) \rightarrow \bot$

   $\vdash \Diamond_x \varphi(\bar{y}) \rightarrow \varphi(\bar{y})$, provided that x is not among the $\bar{y}$

3. Suppose that $\vdash \varphi \rightarrow \psi$ with x not free in ψ:

   Then:

   $\neg\psi \vdash \neg\varphi$

   $\neg\psi \vdash \varphi \rightarrow \bot$

   $\neg\psi \vdash \Diamond_x \varphi \rightarrow \bot$

   $\neg\psi \vdash \neg\Diamond_x \varphi$

whence $\vdash \Diamond_x \varphi \to \psi$.

4. An application of (3) is:

$$\vdash \Diamond_x \varphi \to \Diamond_x \varphi$$
$$\vdash \Diamond_x \Diamond_x \varphi \to \Diamond_x \varphi$$

5. Also,

$$\vdash \varphi \to \exists x \varphi$$
$$\vdash \Diamond_x \varphi \to \exists x \varphi$$

6. As a final illustration, we prove a useful principle for later reference, namely: $\vdash \neg\Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x \psi(x,\bar{y}))$:

$$\vdash \psi(z,\bar{y}) \wedge \neg\Diamond_x \psi(x,\bar{y}) \to \psi(z,\bar{y})$$
$$\vdash \Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x \psi(x,\bar{y})) \to \Diamond_z \psi(z,\bar{y})$$
$$\vdash \Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x \psi(x,\bar{y})) \to \Diamond_x \psi(x,\bar{y})$$

and

$$\vdash \psi(z,\bar{y}) \wedge \neg\Diamond_x \psi(x,\bar{y}) \to \neg\Diamond_x \psi(x,\bar{y})$$
$$\vdash \Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x \psi(x,\bar{y})) \to \neg\Diamond_x \psi(x,\bar{y})$$

(the latter step is as in example (3) above). Therefore,

$$\vdash \Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x \psi(x,\bar{y})) \to \bot$$
$$\vdash \neg\Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x \psi(x,\bar{y}))$$

Theorem 1 *The minimal logic is complete for universal validity.*

Proof. By a standard Henkin construction. The key point, as usual, is to create a maximally consistent set of formulas Σ - this time, adding suitable witnesses (new variables) for accepted formulas $\Diamond_x \varphi$:

If Σ_n is consistent with $\Diamond_x \varphi(x,\bar{y})$,

then add a *new* individual variable z with

1. $\varphi(z,\bar{y})$,
2. $\{\psi(z,\bar{y}) \to \Diamond_x \psi(x,\bar{y}) |$ for all formulas $\psi\}$.

Claim. *This extension is consistent.*

Proof. Suppose it were inconsistent. Then, for some fresh variable z and some finite disjunction of formulas ψ_i:

$$\Sigma_n \vdash \varphi(z,\bar{y}) \to \bigvee_i (\psi_i(z,\bar{y}) \wedge \neg\Diamond_x \psi_i(x,\bar{y})).$$

Then also

$$\Sigma_n \vdash \Diamond_z\varphi(z,\bar{y}) \to \bigvee_i \Diamond_z(\psi_i(z,\bar{y}) \wedge \neg\Diamond_x\psi_i(x,\bar{y})).$$

Therefore, since $\vdash \Diamond_x\varphi(x,\bar{y}) \equiv \Diamond_z\varphi(z,\bar{y})$ (by Alphabetic Variants), $\{\Sigma_n,\ \Diamond_x\varphi(x,\bar{y})\}$ must be consistent with some

$$\Diamond_z(\psi_i(z,\bar{y}) \wedge \neg\Diamond_x\psi_i(x,\bar{y})).$$

But this contradicts the earlier derivability of the formula

$$\neg\Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x\psi(x,\bar{y})).$$

□

Now construct the Henkin model as usual, and set

$$R(z,y_1,\dots,y_n) \ \Leftrightarrow_{df}\ \forall\varphi : \varphi(z,\bar{y}) \in \Sigma \Rightarrow \Diamond_x\varphi(x,\bar{y}) \in \Sigma$$

(Note that we could have as well define $R(z,\{y_1,\dots,y_n\})$ in the same way.) This definition may be compared with the usual introduction of the alternative relation R in completeness proofs for Modal Logic. To demonstrate the adequacy of the present Henkin model, all we have to prove is the following decomposition:

$$\Diamond_x\varphi(x,\bar{y}) \in \Sigma \ \text{ iff } \ \exists z : R(z,\bar{y})\ \&\ \varphi(z,\bar{y}) \in \Sigma$$

From left to right, this is guaranteed by the above construction of Σ (through the addition of all formulas of the second kind). From right to left, this is a trivial consequence of the definition of R. □

If we look at the above completeness proof (and earlier examples of derivabilities), we see that no structural contraction rule or ordinary quantifier rules have been used. This observation (which is quite analogous with the situation in the minimal modal logic) motivates the conjecture the minimal logic without ordinary quantifiers is decidable. Indeed, the following theorem holds

Theorem 2 *The minimal logic without ordinary quantifiers and without equality is decidable.*

Proof. This is shown in Alechina (1994).

3 Model Theory

Now, to illustrate the semantical properties of modal quantifiers, we shall consider an analogue to the basic model-theoretic invariance relation of modal logic. In what follows, we talk about the language $EL(\Diamond)$ (without ordinary quantifiers).

Definition 2 *A* bisimulation $\mathcal{B}$ *between two models* $M_1 =< D_1, R_1, V_1 >$ *and* $M_2 =< D_2, R_2, V_2 >$ *is a family of partial isomorphisms* π *with the following properties:*

1 π *is a partial bijection with* $dom(\pi) \subseteq D_1$ *and* $ran(\pi) \subseteq D_2$*;*

2 If $\{d_1,\dots,d_n\} \subseteq dom(\pi)$*, then for all predicate letters*

$$< d_1,\dots,d_n > \in V_1(P^n) \ \Leftrightarrow < \pi(d_1),\dots,\pi(d_n) > \in V_2(P^n)$$

($d_1,\dots,d_n$ are not necessarily distinct).

3a If $D \subseteq dom(\pi)$ and $R_1(d, D)$, then there exists an element d' in D_2 such that $R_2(d', \pi[D])$ and $\{< d, d' >\} \cup \ \pi \in \mathcal{B}$.

3b If $D' \subseteq ran(\pi)$, $D' = \pi[D]$, and $R_2(d', D')$, then there exists an element d in D_1 such that $R_1(d, D)$ and $\{< d, d' >\} \cup \ \pi \in \mathcal{B}$. [1]

Invariance Lemma *If φ is a formula of $EL(\Diamond)$ with the set of free variables $VAR(\varphi) \subseteq \{y_1, \ldots, y_n\}$, M_1 and M_2 are bisimilar models, and for all y_i ($1 \leq i \leq n$) $v_1(y_i) \in dom(\pi)$ and $v_2(y_i) = \pi(v_1(y_i))$, $\pi \in \mathcal{B}$, then*

$$M_1, v_1 \models \varphi \ \Leftrightarrow \ M_2, v_2 \models \varphi$$

Proof. By induction on the length of φ.

- φ is a k-place predicate letter. - By clause (2) in the definition of bisimulation.
- $\varphi = (x = y)$. $M_1, v_1 \models x = y$ if and only if $v_1(x) = v_1(y)$. Since π is a function, and $v_1(x) \in dom(\pi)$, $\pi(v_1(x)) = \pi(v_1(y))$, that is, $v_2(x) = v_2(y)$ and $M_2, v_2 \models x = y$. Backwards: the same argument, using the fact that π is a bijection.
- $\varphi = \neg\psi$: by the inductive hypothesis,

  $$M_1, v_1 \models \psi \ \Leftrightarrow \ M_2, v_2 \models \psi$$

  and hence

  $$M_1, v_1 \models \neg\psi \ \Leftrightarrow \ M_2, v_2 \models \neg\psi.$$

- $\varphi = \psi_1 \wedge \psi_2$. Again, by the inductive hypothesis,

  $$M_1, v_1 \models \psi_1 \ \Leftrightarrow \ M_2, v_2 \models \psi_1$$

  $$M_1, v_1 \models \psi_2 \ \Leftrightarrow \ M_2, v_2 \models \psi_2$$

  and so,

  $$M_1, v_1 \models \psi_1 \wedge \psi_2 \ \Leftrightarrow \ M_2, v_2 \models \psi_1 \wedge \psi_2$$

- $\varphi = \Diamond_x \psi(x, \bar{y})$. Assume $M_1, v_1 \models \Diamond_x \psi(x, \bar{y})$. By the semantic truth definition, there exists an assignment v'_1 which differs from v_1 at most in its assignment of value to x, such that $R(v'_1(x), v'_1(\bar{y}))$ and $M_1, v'_1 \models \psi(x, \bar{y})$. By assumption, $y_1, \ldots, y_n \in dom(\pi)$. By clause 3a, there is $d' \in D_2$ with $R(d', \pi v'_1(\bar{y}))$, i.e. $R(d', v_2(\bar{y}))$ (since v'_1 and v_1 agree on $\bar{y}$), and $\{< d, d' >\} \cup \ \pi \in \mathcal{B}$. Put $v'_2 =_x v_2$, $v'_2(x) = d'$. Then, for the $\pi' \in \mathcal{B}$ which consists of π and the pair $< d, d' >$, $v'_2(x) = \pi'(v'_1(x))$, and for all y_i, $v'_2(y_i) = \pi(v'_1(y_i))$.

  By the inductive hypothesis, $M_2, v'_2 \models \psi(x, \bar{y})$. But then $M_2, v_2 \models \Diamond_x \psi(x, \bar{y})$. The same argument works backwards. $\Box$

Continuing the analogy with modal logic, we define a translation of $EL(\Diamond)$ formulas into the appropriate first-order logic, which is our original base language enriched with a dependence predicate R. The *standard translation ST* is defined as follows:

[1] Alternatively, we could restrict clause 3 to R-successors of the whole domain and range, while adding a further clause closing $\mathcal{B}$ under restrictions.

- $ST(P_i^n(t_1 \dots t_n)) := P_i^n(t_1 \dots t_n)$;
- $ST(t_1 = t_2) := (t_1 = t_2)$;
- ST commutes with classical connectives;
- $ST(\Diamond_x \varphi(x, \bar{y})) := \exists x(R(x, \bar{y}) \wedge ST(\varphi(x, \bar{y})))$.

Claim 1 *If φ is a formula of $EL(\Diamond)$, then*

$$M, v \models \varphi \;\Leftrightarrow\; M', v \models ST(\varphi),$$

for the classical model $M' = (D, V')$, where V' extends V to interpret the predicate R as R_M.

Definition 3 *The* modal formulas *(being those formulas which are standard translations of $EL(\Diamond)$ formulas) are the least set X of first-order formulas such that*

- *atomic formulas belong to X,*
- *if ψ_1 and ψ_2 are in X, then so are $\neg\psi_1$ and $\psi_1 \wedge \psi_2$,*
- *if $\varphi(x, \bar{y}) \in X$, then $\exists x(R(x, \bar{y}) \wedge \varphi(x, \bar{y}))$ is in X too.*

Theorem 3 *A first-order formula φ is equivalent to a modal formula if and only if it is preserved under bisimulation.*

Proof. The direction from left to right follows from Invariance Lemma above. For the converse, let φ be a first-order formula with variables $x_1, \dots, x_n$, preserved under bisimulation. We want to prove that it is equivalent to a modal formula.

Define the set $CONS_\Diamond(\varphi)$ as $\{\alpha : \alpha$ is a modal formula, $\varphi \models \alpha$ and the free variables of α are among $x_1, \dots, x_n\}$. If we can prove that

$$(*) \qquad CONS_\Diamond(\varphi) \models \varphi,$$

then we are done. For, by compactness, there will be some finite subset $\alpha_1, \dots, \alpha_m$ of $CONS_\Diamond(\varphi)$ with $\alpha_1, \dots, \alpha_m \models \varphi$. By definition, $\varphi \models \alpha_1, \dots, \alpha_m$. So, then φ is equivalent to $\alpha_1 \wedge \dots \wedge \alpha_n$, which is a conjunction of standard translations of $EL(\Diamond)$ formulas, i.e. a standard translation of the conjunction of those formulas.

Now we start proving $(*)$. Assume that for some model $M, v \models CONS_\Diamond(\varphi)$. We show that $M, v \models \varphi$. Let us denote the set of all modal formulas true in M and having free variables among $x_1, \dots, x_n$ as X_M. This is consistent with φ: for, if it is not, there is a finite set $\psi_1, \dots, \psi_k$ of formulas from X_M, such that $\bigwedge_i \psi_i \to \neg\varphi$. Then $\varphi \to \bigvee_i \neg\psi_i$. But $\bigvee_i \neg\psi_i$ is a modal formula (if every ψ_i is). Since it is a consequence of φ, it must be true in X_M. A contradiction.

Therefore there should be a model N for $\varphi \cup X_M$: say, $N, v' \models \varphi \cup X_M$.

Let $v(x_1) = d_1, \dots, v(x_n) = d_n$ in M and $v'(x_1) = d'_1, \dots, v'(x_n) = d'_n$ in N. Now, take ω-saturated elementary extensions $\mathcal{M}$ and $\mathcal{N}$ of M and N. We define a relation of bisimulation between $\mathcal{M}$ and $\mathcal{N}$ as follows:
(**) $\quad \mathcal{B}$ is the family of partial mappings π such that $\pi = \{(e_1, \pi(e_1)), \dots, (e_n, \pi(e_n))\}$ if

for all modal formulas ψ with at most free variables $x_1, \ldots, x_n$ and any two assignments v, v' with $v(x_i) = e_i$, $v'(x_i) = \pi(e_i)$ $(1 \leq i \leq n)$,

$$\mathcal{M}, v \models \psi \;\Leftrightarrow\; \mathcal{N}, v' \models \psi$$

To prove that (**) indeed defines a bisimulation relation, we must check that the properties (1)–(3) hold for $\mathcal{B}$. Here, (1) is trivial. Case (2) is immediate, since atomic formulas are also standard translations of (atomic) formulas in $EL(\diamond)$). Next, we check the zigzag clause 3a. Assume that $e_1, \ldots, e_k \in dom(\pi)$ and $R(e, e_1, \ldots, e_k)$. We must prove that there exists e' in $\mathcal{N}$ such that $R(e', \pi(e_1), \ldots, \pi(e_k))$ and $\{< e, e' >\} \cup \pi \in \mathcal{B}$. Take the set Ψ of all modal formulas with variables interpreted as $e, e_1, \ldots, e_k$ which are true in $\mathcal{M}$ under variable assignment v. We need an element e' in $\mathcal{N}$ such that all formulas in Ψ are true in $\mathcal{N}$ under v' when e' is assigned to the variable which was assigned e in $\mathcal{M}$. By saturation, it suffices to find such an e' for each finite subset Ψ_0 of Ψ. But these must exist, because the modal formula $ST(\diamond_x \bigwedge \Psi_0(x, e_1, \ldots, e_k))$ holds in $\mathcal{M}$ and hence $ST(\diamond_x \bigwedge \Psi_0(x, \pi(e_1), \ldots, \pi(e_k)))$ holds in $\mathcal{N}$. The appropriate check for the converse direction 3b is proved analogously.

Recall that $v(x_i) = d_i$ and $v'(x_i) = d'_i$, $1 \leq i \leq n$. We must also show that $\{< d_1, d'_1 >, \ldots, < d_n, d'_n >\} \in \mathcal{B}$. But this is so because for all modal formulas ψ with variables interpreted as $d_1, \ldots, d_n$ in M,

$$M, v \models \psi \;\Leftrightarrow\; N, v' \models \psi$$

(by the construction of N), and hence

$$\mathcal{M}, v \models \psi \;\Leftrightarrow\; \mathcal{N}, v' \models \psi.$$

Finally, since φ is invariant under bisimulation and $\{< d_1, d'_1 >, \ldots, < d_n, d'_n >\} \in \mathcal{B}$, $\mathcal{N} \models \varphi(d'_1, \ldots, d'_n)$ will now imply $\mathcal{M} \models \varphi(d_1, \ldots, d_n)$. Since $\mathcal{M}$ is an elementary extension of M, $M \models \varphi(d_1, \ldots, d_n)$, that is, $M, v \models \varphi(x_1, \ldots, x_n)$, and we are done. □

To conclude this section, we add some remarks about preservations properties for the full $EL(\exists, \diamond)$ language. Since it includes the whole first-order language, bisimulation is obviously not enough to preserve all formulas:

Claim 2 *If φ does contain $\forall$ or $\exists$, bisimulation does not preserve truth.*

Proof. Let M_1 and M_2 be as follows:

$M_1 =< D_1, R_1, V_1 >$: $D_1 = \{d, d'\}$, $R_1 = \emptyset$, $V_1(P) = \{< d, d' >\}$;

$M_2 =< D_2, R_2, V_2 >$: $D_2 = \{e\}$, $R_2 = \emptyset$, $V_2(P) = \emptyset$.

Then $M_1, [d/x] \models \exists y P(x, y)$ and $M_2, [e/x] \not\models \exists y P(x, y)$. But at the same time, a bisimulation between these two models exists: $\mathcal{B} = \{< d, e >\}$. □

One can strengthen the above notion of bisimulation to preserve the full language, much as happens in modal logic extended with a "universal modality". The result is essentially the standard notion of "partial isomorphism" $\cong_p$ from abstract model theory.

4 Frame Correspondence

In this section and in the following one we extend the standard translation to $EL(\exists, \Diamond)$. An extra clause for ordinary quantifiers has to be added; as it is to be expected, ST commutes with ordinary quantifiers.

If a formula φ of $EL(\exists, \Diamond)$ is valid in a frame F (under an assignment v), then classically

$$F, v \models \forall P_i^n \dots \forall P_l^m ST(\varphi),$$

where $P_i^n, \dots, P_l^m$ are the predicate letters in φ. If this second-order formula has a first-order equivalent (containing only R and $=$), φ is called *first-order definable.* This means that if φ is true in all models over F, then R has the property defined by φ, and vice versa. Additional quantifier principles added to the minimal logic will now express special conditions on the relation R. One bunch of examples arises if we look at some properties of the standard existential quantifier $\exists$:

Unrestricted Distribution $\quad \Diamond_x(\varphi \vee \psi) \leftrightarrow \Diamond_x\varphi \vee \Diamond_x\psi.$

In one direction, this gives us unrestricted "Monotonicity" for $\Diamond$:

$$\Diamond_x\varphi \to \Diamond_x(\varphi \vee \psi).$$

This corresponds to the frame condition of

Upward Monotonicity $\quad R(x, \bar{y}) \to R(x, \bar{y}\bar{z})$

Proof. Suppose that $R(x, \bar{y})$. Define the following predicate:

$$P(u, \bar{v}) := \; u = x \;\wedge\; \bar{v} = \bar{y}.$$

We have $R(x, \bar{y}) \wedge P(x, \bar{y})$, whence $\Diamond_x P(x, \bar{y})$ holds. Therefore,

$$\Diamond_x(P(x, \bar{y}) \vee \bot(\bar{z}))$$

(where $\bot(\bar{z})$ is any contradiction involving $\bar{z}$): i.e., there exists d with $R(d, \bar{y}, \bar{z})$ and $P(d, \bar{y}) \vee \bot(\bar{z})$: the latter must be because $P(d, \bar{y})$: i.e. $d = x$, and hence $R(x, \bar{y}, \bar{z})$. $\Box$

By a similar kind of argument, again making an appropriate substitution for the two predicates involved, the opposite direction

$$\Diamond_x(\varphi \vee \psi) \to \Diamond_x\varphi \vee \Diamond_x\psi$$

corresponds to the frame condition of

Downward Monotonicity $\quad R(x, \bar{y}\bar{z}) \to R(x, \bar{y})$

Together, these reduce the finitary relation R to an essentially *unary* "restriction" to the subdomain of all objects d satisfying the condition $R(d)$. It would also be of interest to see whether we can stop short of this, with quantifiers merely reducing the finitary relation R to a compound of *binary* ones (as happens in the generalized modal semantics for program operators proposed in van Benthem 1992).

Remark. Classical analogies may be slightly misleading here. E.g., the implication

$$\Diamond_x(\varphi \wedge \psi) \to \Diamond_x\varphi$$

(cf. $\Diamond(\varphi \wedge \psi) \rightarrow \Diamond\varphi$) expresses Downward Monotonicity, rather than the Upward Monotonicity of

$$\Diamond_x\varphi \rightarrow \Diamond_x(\varphi \vee \psi)$$

(cf. $\Diamond\varphi \rightarrow \Diamond(\varphi \vee \psi)$), even though the latter is equivalent with it in standard modal logic. Thus, it should in fact imply unlimited distribution - as may be seen using the available distribution in our minimal logic. In the latter calculus, "limited distribution" sanctions

1. $\Diamond_x(\varphi(x,\bar{y}) \vee \psi(x,\bar{z})) \rightarrow \Diamond_x((\varphi(x,\bar{y}) \wedge \top(\bar{z})) \vee (\psi(x,\bar{z}) \wedge \top(\bar{y})))$
2. $\Diamond_x((\varphi(x,\bar{y}) \wedge \top(\bar{z})) \vee (\psi(x,\bar{z}) \wedge \top(\bar{y}))) \rightarrow$
   $\rightarrow \Diamond_x(\varphi(x,\bar{y}) \wedge \top(\bar{z})) \vee \Diamond_x(\psi(x,\bar{z}) \wedge \top(\bar{y}))$,
3. from which the unlimited version $\Diamond_x(\varphi \vee \psi) \rightarrow \Diamond_x\varphi \vee \Diamond_x\psi$ follows by the above implication, passing to the appropriate conjuncts. □

Finally, the above unary relation gets trivialized to universality by the principle of

Instantiation $\qquad \varphi \rightarrow \Diamond_x\varphi$

This corresponds to the frame condition $\forall x\forall y R(x,y)$ (provided that we assume non-empty individual domains, that is). The idea is this: let x,y be arbitrary, and let $P(x,y)$ hold of just these. We must have that $\Diamond_x P(x,y)$: i.e., some object d exists with $R(d,y)$ and $P(d,y)$: whence $R(x,y)$. □

Another source of examples is the analysis of various properties of the standard quantifier $\exists$ which are all lumped together as being "valid" in ordinary predicate logic, but which now become distinguishable as different properties of dependence. To be sure, such differences also become visible in other more sensitive semantics, such as those for intuitionistic predicate logic, or the logic of polyadic generalized quantifiers. Indeed, one concrete interpretation of the above structured domains would be the following:

individuals	pairs <world, individual>
dependence	$(w,x)R(v,y)$ iff $w \subseteq v$ & $y = x$,

inspired by standard possible worlds semantics for intuitionistic logic. We continue with one example of this kind:

Prenex operations $\qquad \Diamond_x(\varphi \vee \psi) \leftrightarrow \varphi \vee \Diamond_x\psi$, where x not free in φ.

The direction $\rightarrow$ here turns out universally valid in case $\psi, \varphi\vee\psi$ have the same free variables, and hence derivable:

$\neg\varphi \vdash (\varphi \vee \psi) \rightarrow \psi$

$\neg\varphi \vdash \Diamond_x(\varphi \vee \psi) \rightarrow \Diamond_x\psi$

$\vdash \Diamond_x(\varphi \vee \psi) \rightarrow \Diamond_x\psi \vee \varphi$, Otherwise, it will enforce the earlier Downward Monotonicity: $R(x, \bar{y}\bar{z}) \to R(x, \bar{y})$. The direction $\leftarrow$ corresponds to the conjunction of $R(x, \bar{y}) \to R(x, \bar{y}\bar{z})$ and $\exists x R(x, \bar{y})$. $\Box$

A comparison of quantifier axioms and similar modal axioms can also provide some interesting correspondences. For example, how would one write a quantifier version of the well-known K4-axiom: as

$$\Diamond_x \varphi \to \Diamond_x \Diamond_x \varphi$$

or with the more complex decoration

$$\Diamond_x \varphi \to \Diamond_y \Diamond_x \varphi?$$

The first one is universally valid, the second one defines

$$R(x, y\bar{z}) \to R(y, \bar{z})$$

(in case y is free in φ). Another direction is also possible: which quantifier principles correspond to well known properties of Kripke frames? Well-known examples are the three defining properties of equivalence relations:

Reflexivity $\quad R(x, x)$

Transitivity $\quad R(y, x) \wedge R(z, y) \to R(z, x)$

Symmetry $\quad R(x, y) \to R(y, x)$

Fact. *The following principles are definable in $EL(\exists, \Diamond)$:*

- *Reflexivity corresponds to $\Diamond_y x = y$;*
- *Transitivity corresponds to $\Diamond_y(\top(x) \wedge \Diamond_z(\top(y) \wedge P(z))) \to \Diamond_z(\top(x) \wedge P(z))$*
- *Symmetry corresponds to $\forall x \Box_y P(x, y) \to \forall y \Box_x P(x, y)$.*

(Proofs will be given in Section 5 below.)

Some negative results concerning definability of first-order properties in $EL(\Diamond)$ alone can be obtained using frame constructions familiar from modal logic.

Definition 4 *Let $F = < D, R >$ be a frame and $d_1, \ldots, d_n \in D$. A subframe $F' = < D', R' >$ of F is* generated *by $d_1, \ldots, d_n$ if*

- *D' is the smallest subdomain of D containing $d_1, \ldots, d_n$ which is closed under accessibility, and*
- *R' is the restriction of R to D'.*

Theorem 4 *Let F' be a generated subframe of F, v a valuation restricted to the elements of D', and φ a formula of $EL(\Diamond)$. Then*

$$F, v \models \varphi \;\Leftrightarrow\; F', v \models \varphi$$

(in other words, $EL(\Diamond)$-formulas are invariant for generated subframes).

Proof. For any pair of models $M =< F, V >$ and $M' =< F', V >$ the identity map from D' to D gives an obvious bisimulation, and we can apply our invariance results from Section 3. $\square$

Examples (Modal undefinability).

- $\exists x \neg R(x,x)$ is not definable by an $EL(\Diamond)$ formula. Consider

  $$F = < \{d_1, d_2\}, \{< d_1, d_1 >\} >,$$

  where it holds, and the generated subframe

  $$F' = < \{d_1\}, \{< d_1, d_1 >\} >,$$

  where it fails.

- $\forall x \forall y (x \neq y \ \Rightarrow R(x,y))$ is not definable in $EL(\Diamond)$. Consider the same two frames, but now in the opposite direction. $\square$

The language of $EL(\exists, \Diamond)$ with ordinary quantifiers added is much more powerful. For instance, $\exists x \neg R(x,x)$ is definable as $\exists x \neg \Diamond_y x = y$, and
$\forall x \forall y (x \neq y \Rightarrow R(x,y))$ as $\exists y (x \neq y \wedge P(x,y)) \rightarrow \Diamond_y P(x,y)$. Of course, a great deal of expressive power is due to the presence of identity in this language. Here is a more general result demonstrating this.

Theorem 5 *Every purely universal R-condition is* $\Diamond, \exists$*-definable.*

Proof. (Cf. Proposition 2.4 in de Rijke (1992a)). Consider any R-condition of the following form:

$$\forall y_1 \ldots \forall y_n \ BOOL(R, =, y_1, \ldots, y_n),$$

where "$BOOL$" is a purely Boolean condition. Introduce a predicate P_{y_i} for every universally quantified variable y_i, which holds exactly for y_i: $\exists ! x P_{y_i}(x)$. Define a translation * of first-order formulas with R into $EL(\exists, \Diamond)$, such that * commutes with Boolean connectives and $=$, where

$$(R(y, \bar{z}))^* = \ \Diamond_u (P_y(u) \wedge \top(\bar{z})).$$

Then the $EL(\exists, \Diamond)$ equivalent of the R-property will be

$$\exists ! x P_{y_1}(x) \wedge \ldots \wedge \exists ! x P_{y_n}(x) \rightarrow (BOOL(R, =, y_1, \ldots, y_n))^*$$

$\square$

Open problem *Are all first-order properties of R* $\exists, \Diamond$*-definable?*

We conjecture that the answer to this question is negative. A possible counterexample is the first-order formula $\exists x \forall y R(x,y)$.

There is a more general theory behind these various observations. The above axioms whose frame correspondences were analysed all had "Sahlqvist forms" in a suitably general sense, and the proof method depends on finding suitable "minimal substitutions". In the next section, we make this precise.

For now, we conclude with another aspect of modal frame correspondence. It is known that the question whether a modal axiom corresponds to a first-order condition on frames is undecidable (Chagrova (1991)). One would expect that the same holds for relational generalized quantifiers. And indeed we have this

Proposition. First-order correspondence for $EL(\Diamond)$ formulas is undecidable.

Proof. The idea is as follows. Let φ be a modal formula. It defines a first-order condition on frames if and only if $\forall P_1 \dots \forall P_n ST(\varphi)$ has a first-order equivalent, where $P_1, \dots, P_n$ are all predicate symbols in $ST(\varphi)$ and $ST(\varphi)$ is the standard translation of φ in the first-order language. Analogously for the generalized quantifier formulas. We find an effective fragment of the $EL(\Diamond)$ language whose first-order translations are effectively equivalent to the standard translations of modal formulas. Thus, a modal formula is first-order definable iff its $EL(\Diamond)$-counterpart is, and hence the correspondence problem for generalized quantifiers (in the latter language) is undecidable.

Consider the following translation $(\ )^i$ taking modal formulas to formulas of $EL(\Diamond)$ with one free variable w_i:

- $(p_n)^i = P_n(w_i)$
- commute with the Booleans
- $(\Box\varphi)^i = \Box_{w_{i+1}}(\top(w_i) \wedge (\varphi)^{i+1})$

The only thing to prove is that for every modal formula φ, $ST(\varphi)$ is provably equivalent in first-order logic to the standard translation of φ^0. (There is a minor difference which does not influence the result: modal R is the converse of our R.)

This may be shown by induction on complexity of φ. Let φ be a propositional variable. Then $ST(\varphi)$ is an atomic formula which is a standard translation of $(\varphi)^0$. In case φ is a negation or a conjunction, apply the inductive hypothesis (both standard translations commute with the Booleans). Let $\varphi = \Box\psi$. $ST(\Box\psi)[w] = \forall w'(R(w, w') \rightarrow ST(\psi)[w']) = \forall w'(R(w, w') \rightarrow \top(w) \wedge ST(\psi)[w']) =$ (by the inductive hypothesis) $= ST(\Box_{w'}(\top(w) \wedge ST(\psi)[w'])$. $\Box$

5 A Sahlqvist Theorem

Theorem 6 *All formulas of the "Sahlqvist form" $\bigwedge_i Qu_1 \dots Qu_k(\varphi \rightarrow \psi)$, where Qu_j is either $\forall u_j$ or $\Box_{u_j}$, and*

1. *φ is constructed from*
   - *atomic formulas, possibly prefixed by $\Box_x$, $\forall$;*
   - *formulas in which predicate letters occur only negatively*

   using $\wedge, \vee, \Diamond_x, \exists$
2. *in ψ all predicate letters (except $=$) occur only positively*

are first-order definable.

Proof. If every conjunct is first-order definable, the whole conjunction is. Therefore without loss of generality we can concentrate on a formula of the form $Qu_1 \dots Qu_k(\varphi \to \psi)$. First we translate it into second-order logic:

$$\forall P_1^n \dots \forall P_l^m \forall u_1 \dots \forall u_k(\mathcal{R} \wedge ST(\varphi) \to ST(\psi)),$$

where $P_1^n \dots P_l^m$ are all the predicates in $\varphi \to \psi$ and $\mathcal{R}$ is a conjunction of R-statements corresponding to the $\Box$-quantifiers in the prefix. Then we remove all "empty" quantifiers (those binding variables not occurring in their scope), and rename bound individual variables in such a way that every quantifier gets its own variable which is distinct from any free variable occurring in the formula. Now it is possible to move all existential quantifiers occurring in positive subformulas of $ST(\varphi)$ to a prefix, using the following equivalences:

$$\exists x A(x) \vee \exists y B(y) \equiv \exists x \exists y(A(x) \vee B(y))$$

$$\exists x A(x) \wedge B \equiv \exists x(A \wedge B)$$

with the usual provisos on freedom and bondage. $ST(\varphi)$ has now been rewritten as

$$\exists y_1 \dots \exists y_m \varphi'.$$

Since ψ does not contain $y_1, \dots, y_m$ free, $\forall u_1 \dots \forall u_k(\mathcal{R} \wedge ST(\varphi) \to ST(\psi)$ is equivalent to

$$\forall x_1 \dots \forall x_n(\mathcal{R} \wedge \varphi' \to ST(\psi)),$$

where $x_1, \dots, x_n$ include $\bar{u}$ and $\bar{y}$.

Next, it would be convenient to get rid of the disjunctions in φ'. Let $\varphi' \equiv \phi_1 \vee \phi_2$.

$$\forall x_1 \dots \forall x_n((\mathcal{R} \wedge \phi_1) \vee (\mathcal{R} \wedge \phi_2) \to ST(\psi))$$

is equivalent to

$$\forall x_1 \dots \forall x_n(\mathcal{R} \wedge \phi_1 \to ST(\psi)) \wedge \forall x_1 \dots \forall x_n(\mathcal{R} \wedge \phi_2 \to ST(\psi)).$$

We can restrict attention to one of these conjuncts (if both components have a first-order equivalent, then so has their conjunction). So, assume that there are no disjunctions in the antecedent. Thus, we have a formula

$$\forall P_1^n \dots \forall P_l^m \forall x_1 \dots \forall x_n(\varphi' \to ST(\psi)),$$

where $P_1^n \dots P_l^m$ are all the predicates in $\varphi' \to ST(\psi)$, and φ' is a conjunction of "blocks" which are of one of the following forms:

1. standard translations of atomic formulas possibly preceded by universal and $\Box$-quantifiers,
2. R-statements,
3. formulas in which all predicate letters occur only negatively.

Next we rule out the use of negative formulas. The point is that $\varphi' \to ST(\psi)$ can always be rewritten as an implication whose antecedent does not contain negative formulas. Let $\varphi' = \phi_1 \wedge \phi_2$, where ϕ_2 is a negative formula. Then

$$\phi_1 \wedge \phi_2 \to ST(\psi)$$

is equivalent to

$$\phi_1 \to \neg\phi_2 \vee ST(\psi),$$

whose consequent contains only positive occurrences of predicate letters.

Let us denote the antecedent obtained (without negative formulas) φ^*. We shall now define the notion of *a minimal substitution* for every predicate letter in φ^*.

A predicate letter P_i^n can occur in φ^* more than once. Consider an occurrence $\bar{P}_i^n$ of P_i^n in φ^*. First we have to classify the variables of this occurrence (this is the only part where the present proof becomes different from the modal case). Let us assume that

- the variables which stand at the places $i_1, \ldots, i_m$ *in this occurrence* are existentially bound or free; let us denote them $x_1, \ldots, x_m$;

- the variables at the places $j_1, \ldots, j_k$ are universally bound by quantifiers which correspond to $\Box$-quantifiers in the original formula; let us call them $z_1, \ldots, z_k$;

- the rest of the variables is bound by ordinary universal quantifiers; let us call them $v_1, \ldots v_l$.

Before defining a minimal substitution we have to define the notion of an "R-condition" corresponding to the variable z_i:

1. Let $\Box_{z_1}$ be the first (leftmost) generalized quantifier in the sequence of quantifiers preceding $\bar{P}_i^n$, and before $\Box z_1$ the ordinary universal quantifiers $\forall v_1, \ldots, \forall v_s$ occur. Then the R-condition corresponding to z_1 will be $R(z_1, v_1, \ldots, v_s, \bar{x})$,

2. Let $\Box_{z_i}$ be the generalized quantifier following $\Box_{z_{i-1}}$ in our sequence (with some $\forall v_p, \ldots, \forall v_r$ possibly standing in between):

$$\ldots \Box_{z_{i-1}} \forall v_p \ldots \forall v_r \Box_{z_i} \ldots \bar{P}_i^n$$

   If the condition corresponding to z_{i-1} was $R(z_{i-1}, \bar{y})$, then the condition corresponding to z_i is $R(z_i, v_p, \ldots, v_r, z_{i-1}, \bar{y})$.

The minimal substitution $Sb(\bar{P}_i^n)$ for the occurrence of P_i^n in φ^* described above will be:

$P_i^n(u_1, \ldots u_n)$ is the conjunction of

1. $u_{i1} = x_1, \ldots, u_{im} = x_m$;

2. $\top(v_1), \ldots, \top(v_l)$;

3. $R(u_{\alpha_1}, \ldots, u_{\alpha_f})$, where $u_{\alpha_1}, \ldots, u_{\alpha_f}$ are the variables standing at the places $\alpha_1, \ldots, \alpha_f$, and in φ^* for these variables some R-condition (corresponding to one of the variables $z_1, \ldots, z_k$) hold.

Finally, we define

$$Sb(P_i^n, \varphi^*) = \bigvee Sb(\bar{P}_i^n)$$

for all occurrences of P_i^n in φ^*. [2]

[2]Note that we do not need existential quantifiers here to deal with iterations of $\Box$, as in modal logic; instead of $R^n(x, y)$, which is short for $\exists y_1(R(x, y_1) \wedge \ldots \wedge \exists y_{n-1} R(y_{n-1}, y))$, we have, for iterated modalities, $R(y_1, x) \wedge \ldots \wedge R(y, y_{n-1}, \ldots, y_1, x)$.

The result of substituting $Sb(P_i^n, \varphi^*)$ in $\forall x_1 \dots \forall x_m(\varphi^* \to \psi')$, which we shall denote as

$$\forall x_1 \dots \forall x_m(s(\varphi) \to s(\psi))$$

is our intended first-order equivalent, which contains no predicate symbols other than R and $=$. It is easy to see that it follows from the original Sahlqvist axiom, being an instantiation of a universal second-order formula

$$\forall P_i^n \dots \forall P_l^m \forall x_1 \dots \forall x_m(\varphi^* \to \psi').$$

We must prove the other direction to have an equivalence.

Assume that $\forall x_1 \dots \forall x_m(s(\varphi) \to s(\psi))$ holds in some frame F under a variable assignment v. Assume, for some interpretation function V, that φ^* holds in $M =< F, V >$. To show that ψ' holds in the same model, we need the following two assertions:

Lemma 1 *For all M, v: $M, v \models \varphi^* \;\Rightarrow\; M, v \models s(\varphi)$*

Lemma 2 *Let $M, v \models \varphi^*$, and let $v(x_1) = d_1, \dots, v(x_m) = d_m$. Define $V^*(P_i^n)$ as the set of all n-tuples which satisfy $Sb(P_i^n, \varphi^*)$ under v (that is, with $d_1, \dots, d_m$ assigned to $x_1, \dots, x_m$). Then*

$$V^*(P_i) \subseteq V(P_i).$$

From the first lemma it follows that $s(\varphi)$ also holds for V and v; and hence $s(\psi)$ holds. Since ψ' is positive, Lemma 2 (with the Monotonicity Lemma for classical logic) implies that $M, v \models \psi'$, as was to be shown.

Proof of lemma 1 φ^* has the form $\Psi \wedge \Gamma \wedge \Theta$, where Ψ is a conjunction of R-statements corresponding to the translations of $\Diamond$-quantifiers, Γ is a conjunction of atomic formulas, and Θ a conjunction of universally bound implications. It is easy to check that the two latter conjuncts turn into tautologies after substituting $Sb(P_i, \varphi^*)$ for every P_i in φ^*. It means that $\vdash\; s(\varphi) \equiv \Psi$, so it follows from any conjunction including Ψ.

Proof of lemma 2 (a.) Consider the case when the occurrence of P_i is in Γ. Every V which makes the formula true under v should include at least one tuple which satisfies the conditions from Ψ. Then it contains the tuple which satisfies $Sb(\bar{P_i})$. (b.) Let $\bar{P_i}$ be in Θ. Then it is of the form

$$\forall y_1 \dots \forall y_{k+l}(\mathcal{R}_1 \wedge \dots \wedge \mathcal{R}_k \to P_i(\bar{y}, \bar{x})),$$

where $\mathcal{R}_1 \dots \mathcal{R}_k$ are the R-conditions corresponding to the generalized quantifiers. If φ^* is true under V and v, then this subformula is true, too, which means that $V(P_i)$ includes at least all tuples $< d_1, \dots, d_n >$ for which the relation R holds between $\alpha_1, \dots, \alpha_f$th members, for each of the k R-conditions. So, again it contains all tuples which satisfy $Sb(\bar{P}_i, \varphi^*)$. But if for every occurrence of P_i, the set of tuples satisfying $Sb(\bar{P_i}, \varphi^*)$ is a subset of $V(P_i)$, then also their union is in $V(P_i)$. Thus, $V^*(P_i) \subseteq V(P_i)$. $\square$

Examples. Here is how the above Sahlqvist algorithm works on the earlier examples of reflexivity, transitivity and symmetry.

- *Reflexivity.* Consider $\Diamond_y x = y$. Its standard translation is

$$\exists y(R(y,x) \wedge x = y),$$

which is equivalent to $R(x,x)$.

- *Transitivity.* The standard translation of

$$\Diamond_y(\top(x) \wedge \Diamond_z(\top(y) \wedge P(z))) \to \Diamond_z(\top(x) \wedge P(z))$$

gives us

$$\forall P[\exists y(R(y,x) \wedge \top(x) \wedge \exists z(R(z,y) \wedge \top(y) \wedge P(z))) \to \exists u(R(u,x) \wedge \top(x) \wedge P(u))]$$

which can be rewritten in accordance with the Sahlqvist algorithm as

$$\forall P \forall y \forall z(R(y,x) \wedge R(z,y) \wedge P(z)) \to \exists u(R(u,x) \wedge P(u)))$$

The minimal substitution for $P(u)$ is $u = z$, so we obtain

$$\forall y \forall z(R(y,x) \wedge R(z,y) \wedge z = z \to \exists u(R(u,x) \wedge u = z),$$

which is a first-order equivalent of transitivity:

$$\forall y \forall z(R(y,x) \wedge R(z,y) \to R(z,x))$$

- *Symmetry.* The formula

$$\forall x \Box_y P(x,y) \to \forall y \Box_x P(x,y)$$

is translated as

$$\forall P(\forall x \forall y(R(y,x) \to P(x,y)) \to \forall y \forall x(R(x,y) \to P(x,y)))$$

The minimal substitution for $P(u,v)$ is $\top(u) \wedge R(v,u)$:

$$\forall x \forall y(R(y,x) \to \top(x) \wedge R(y,x)) \to \forall x \forall y(R(x,y) \to \top(x) \wedge R(y,x))$$

The antecedent becomes trivial:

$$\top \to \forall x \forall y(R(x,y) \to R(y,x))$$

which can again be written more elegantly as

$$\forall x \forall y(R(x,y) \to R(y,x)).$$

So far our illustations concerned finding first-order equivalents of modal formulas. Recall the reverse problem of defining first-order properties of R by modal formulas.

Fact. *All purely universal R-conditions can be defined using Sahlqvist formulas only.*

Proof. We show that the algorithm for defining R-properties in $EL(\exists, \Diamond)$ described in the Theorem 5 produces Sahlqvist formulas. First, $\bigwedge_i \exists! x P_{y_i}(x)$ is a Sahlqvist antecedent: it can be rewritten as

$$\bigwedge_i \exists y_i P_{y_i}(y_i) \wedge \bigwedge_i \forall x \forall z (P_{y_i}(x) \wedge P_{y_i}(z) \rightarrow x = z)$$

In the second conjunct, all predicate letters occur negatively (but when it is moved to the consequent in accordance with the Sahlqvist algorithm, those occurrences become positive).

Next, in the consequent we have $(BOOL(R, =, y_i))^*$, where some predicate letters again can occur negatively. Rewrite it as a conjunction of disjunctions of "atomic" statements $(\Diamond_u(P_{y_i}(u) \wedge \top(\bar{z})))$ and their negations:

$$\Phi \rightarrow \Psi_1 \wedge \ldots \wedge \Psi_n$$

The above expression is equivalent to the following conjunction:

$$(\Phi \rightarrow \Psi_1) \wedge \ldots \wedge (\Phi \rightarrow \Psi_n),$$

where each of Ψ_j's is a disjunction of atomic statements and their negations. Now move negations of atomic statements to the antecedents:

$$\Phi \rightarrow \neg \Diamond_u(P_y(u) \wedge \top(\bar{z})) \vee \Psi \quad \text{becomes} \quad \Phi \wedge \Diamond_u(P_y(u) \wedge \top(\bar{z})) \rightarrow \Psi$$

As a result, there are no negative occurrences of predicate letters in the consequents. □

Example (Symmetry revisited). Here is one more illustration of the preceding technique. Symmetry can be also defined "locally" using

$$P(x) \wedge \neg \exists x'(x' \neq x \wedge P(x')) \wedge Q(y) \wedge \neg \exists y'(y' \neq y \wedge Q(y')) \rightarrow$$

$$\rightarrow \neg \Diamond_u(P(u) \wedge \top(y)) \vee \Diamond_u(Q(u) \wedge \top(x)) :$$

The latter formula becomes

$$P(x) \wedge Q(y) \wedge \exists u(R(u, y) \wedge P(u)) \rightarrow \exists x'(x' \neq x \wedge P(x') \vee \exists y'(y' \neq y \wedge Q(y')) \vee$$

$$\vee \exists v(R(v, x) \wedge Q(v)),$$

or

$$\forall u[P(x) \wedge Q(y) \wedge R(u, y) \wedge P(u) \rightarrow \exists x'(x' \neq x \wedge P(x') \vee \exists y'(y' \neq y \wedge Q(y')) \vee$$

$$\vee \exists v(R(v, x) \wedge Q(v))].$$

The minimal substitutions are as follows:

$P(z) := \ z = x \vee z = u;$

$Q(z) := \ z = y.$

The resulting formula will be then

$$\forall u((x = x \vee x = u) \wedge y = y \wedge R(u, y) \wedge (u = x \vee u = u) \rightarrow \exists x'(x' \neq x \wedge (x' = x \vee x' = u) \vee$$

$$\vee \exists y'(y' \neq y \wedge y' = y) \vee \exists v(R(v, x) \wedge v = y));$$

applying predicate logic gives

$$\forall u(R(u,y) \to \exists x'(x' \neq x \wedge x' = u) \vee R(y,x))$$
$$\forall u(R(u,y) \wedge \forall x'(x' = u \to x' = x) \to R(y,x),$$

which is equivalent to

$$\forall u(R(u,y) \to R(y,u))$$

□

6 Limitative Results

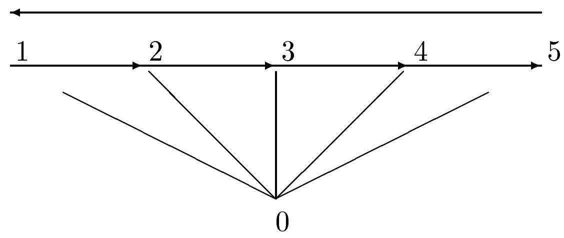
If a formula does not have the form described in our Sahlqvist Theorem, it may lack a first-order equivalent. The proof that a combination $\Box(\ldots \vee \ldots)$ in the antecedent can be fatal, is adapted from the analogous proof for modal logic (see van Benthem 1985, lemma 10.6).

Lemma 3 $\Box_x(\Box_y(P(y) \wedge \top(x,z)) \vee P(x)) \to \Diamond_x(\Diamond_y(P(y) \wedge \top(x,z)) \wedge P(x))$ *is not first-order definable.*

Proof. Define a class of frames F_n as follows:

- $D_n = \{0, 1, \ldots, 2n+1\}$;
- $R_n = \{< i, 0 >: 1 \leq i \leq 2n+1\} \cup \{< i+1, i, 0 >: 1 \leq i \leq 2n, \} \cup \{< 1, 2n+1, 0 >\}$.

Here is a picture illustrating this with $R(j,i,0)$ represented as "there is a line from 0 to i and an arrow from i to j":



For every n and V,

$$F_n, V, [z/0] \models \Box_x(\Box_y(P(y) \wedge \top(x)) \vee P(x)) \to \Diamond_x(\Diamond_y(P(y) \wedge \top(x)) \wedge P(x))$$

Indeed, the antecedent is true if

$$\forall x(R(x,z) \to \forall y(R(y,x,z) \to P(y)) \vee P(x));$$

that is, if for every i with $R(i,0)$ $P(i)$ is true or P holds for each j with $R(j,i,0)$. Each such i has exactly one "successor" j with $R(j,i,0)$ and "predecessor" k with $R(i,k,0)$. They form a chain which has by definition an odd number of members. That is why, if the antecedent is true, then P should hold for some pair of neighbours in this chain. But then the consequent is also true:

$$\exists x(R(x,z) \wedge \exists y(R(y,x,z) \wedge P(y)) \wedge P(x)).$$

Now, assume that our formula had a first-order equivalent. For arbitrary large n, it is consistent with the following set of first-order sentences describing the frames F_n:

$\forall x \forall y (R(x,y) \to \neg R(y,x))$

$\forall x \forall y \forall z (R(x,y,z) \to \neg R(y,x,z))$

$\exists! z \forall y R(y,z)$

$\forall y (\exists! x R(x,y,z) \wedge \exists! u R(y,u,z))$

$\neg \exists x_1 \ldots \exists x_{2n} \exists y (R(x_2, x_1 y) \wedge \ldots R(x_{2n}, x_{2n-1}, y) \wedge R(x_1, x_{2n}, y))$.

The latter formula forbids "loops" of length less than $2n+1$; that is why it is true in F_k for all $k \geq n$.

By compactness, since each finite set of these formulas has a model for suitably large n, they also have a countable model simultaneously. But in all countable models with the above properties (which are isomorphic copies of $\mathbf{Z}$ with ternary R interpreted as $R(j,i,0) := S(j,i)$ and 0 being a fixed element preceding all other elements: $R(i,0)$ for all $i \neq 0$) the formula can easily be refuted by putting $P(i)$ iff $\neg P(i-1)$ and $\neg P(i+1)$. $\square$

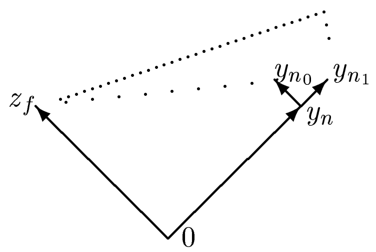
The same result holds for the combination $\Box_x \ldots \Diamond_y$ in the antecedent (the proof is analogous to the proof of lemma 10.2 in van Benthem 1985 for McKinsey axiom):

Lemma 4 $\Box_x \Diamond_y (P(y) \wedge \top(x,z)) \to \Diamond_x \Box_y (P(y) \wedge \top(x,z))$ *does not have a first-order equivalent.*

Proof. Consider the following class of models:

$D = \{0\} \cup \{y_n : n \in N\} \cup \{y_{n_i} : n \in N, i \in \{0,1\}\} \cup \{z_f : f : N \to \{0,1\}\}$;

$R = \{< y_n, 0 >: n \in N\} \cup \{< y_{n_i}, y_n, 0 >: n \in N, i \in \{0,1\}\} \cup \{< z_f, 0 >: f : N \to \{0,1\}\} \cup \{< y_{n_{f(n)}}, z_f, 0 >: n \in N, f : N \to \{0,1\}\}$



(Here an arrow from a to b describes $R(b,a)$, and the combination of arrows from a to b and from b to c - $R(c,b,a)$.)

Any model of this class validates the formula in question: assume

$$M, v = [z/0] \models \Box_x \Diamond_y (P(y) \wedge \top(x,z)).$$

This means that $\forall x (R(x,0) \to \exists y (R(y,x,0) \wedge P(y) \wedge \top(x,0))$ is true, which implies that $\forall n \exists i P(y_{n_i})$ holds. Since for every n either y_{n_0} or y_{n_1} satisfies P, we can choose f such that $P(y_{f(n)})$ for every n. Then the consequent is also true: $\exists x (R(x,0) \wedge \forall y (R(y,x,0) \to (P(y) \wedge \top(x,0)))$ (via $x = z_f$), whence

$$F, v = [z/0] \models \Box_x \Diamond_y (P(y) \wedge \top(x,z)) \to \Diamond_x \Box_y (P(y) \wedge \top(x,z))$$

M is obviously uncountable. Consider any countable elementary submodel M' of F which includes $0, y_n, y_{n_0}, y_{n_1}$ for all n. If our formula had a first-order equivalent, it would be true in M' . But it can be refuted there: since M' is countable, it does not contain some z_f. Put $y_{n_i} \in V(P)$ iff $i = f(n)$. Then the antecedent is still true (all elements which had a successor in P, still have it), but the consequent is false. $\Box$

Another limitation to the above result emerges when we try to obtain its natural generalization towards *completeness* of Sahlqvist logics. Here is a striking problem, due to Michiel van Lambalgen.

Example (Sahlqvist incompleteness).

Consider the following three axioms:

A1. $\Diamond_x x = x$;

A2. $\neg\Diamond_y x = y$;

A3. $\Diamond_x \varphi(x, \bar{y}) \to \Diamond_x(\varphi(x, \bar{y}) \vee \psi(x, \bar{z}))$

These properties are consistent (think of an interpretation for $\Diamond$ like "there exist at least two"). According to the Sahlqvist theorem, these axioms define the following properties of R:

R1. $\exists x R(x)$;

R2. $\neg R(x, x)$;

R3. $R(x, \bar{y}) \to R(x, \bar{y}, \bar{z})$;

But together R1–R3 imply $\perp$:

1. $R(x)$ - R1
2. $R(x) \to R(x, x)$ - R3
3. $R(x, x)$ - 1,2
4. $\neg R(x, x)$ R2
5. $\perp$

This example shows that the match between correspondence and completeness is not as good for modal quantifiers as it is for ordinary modal logic. A natural question arises, whether an analogue of the Sahlqvist's theorem can be proved for *correspondence for completeness.* The answer is given in Alechina and van Lambalgen (1994). In fact, the class of *weak Sahlqvist formulas* which have correspondents in the sense of completeness turns out to be a proper subclass of Sahlqvist formulas: namely, $\Diamond$ and $\exists$ quantifiers are not allowed in the antecedent.

7 Further Directions

In this paper we studied a number of properties of "modal" quantifiers, mostly their model theory and frame correspondence theory. Correspondence in the sense of completeness and its connection to the proof theory of generalized quantifiers is the main topic of Alechina and van Lambalgen (1994).

Another line of research is experimenting further with the truth definition so that to make the quantifier behave even more like a modal operator (and escape the incompleteness phenomenon described above). This can be achieved by replacing our accessibility relation between elements with one between assignments (van Benthem (1994)).

There are interesting connections between both lines of research sketched above and developments in algebraic logic (cf. Nemeti (1993)), namely in cylindric algebras. "Logically", cylindric algebras correspond to first order models with restricted sets of possible assignments. In van Benthem (1994) a set of conditions is found under which "abstract" frames for modal quantifiers (with the new truth definition) can be represented as "assignment frames", i.e. frames of such models. These and similar connections with algebraic logic form one of our directions of further research.

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